Indian Statistical Institute, Bangalore
B. Math. Third Year, Second Semester
Analysis IV

Mid-term Examination Date: 23-02-2016

Maximum marks: 100 Time: 3 hours

In the following the usual metric $d(x,y) = |x-y|$ is considered on $\mathbb{N}, \mathbb{R}, [0,1]$ etc., unless specified otherwise.

(1) Let B be the space of binary sequences:
$$B = \{(b_1, b_2, b_3, \ldots) : b_i \in \{0,1\} \ \forall i \geq 1\}.$$
Define $m : B \times B \to \mathbb{R}$ by
$$m(b,c) = \begin{cases} \frac{1}{2^k} & \text{if } b \neq c \text{ and } k = \min\{j : b_j \neq c_j\} \\ 0 & \text{if } b = c. \end{cases}$$
where $b = (b_1, b_2, b_3, \ldots), c = (c_1, c_2, c_3, \ldots)$.
(i) Show that m is a metric on B; (ii) Show that (B, m) is separable; (iii) Prove or disprove completeness of (B, m); (iv) Write down an isometry from (B, m) to l^∞. (Recall that l^∞ is the sequence space with $d(x,y) = \sup_i |x_i - y_i|$.) [30]

(2) Suppose $f : [0,1] \to \mathbb{R}$ is a function. Then the graph of f is defined as the subset $G(f) = \{(x, f(x)) : x \in [0,1]\}$ of $\mathbb{R}^2$. Give an example of f where it's graph is closed but not bounded. Show that f is continuous if and only if $G(f)$ is closed and bounded (with usual metric on $\mathbb{R}^2$). [15]

(3) Let $g : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}, n \geq 1$ be the functions $g(x) = x \ \forall x$ and $g_n(x) = x + \frac{1}{n} \ \forall x$. Show that as n tends to infinity $\{g_n\}_{n \geq 1}$ converges uniformly to g, but $\{g_n^2\}_{n \geq 1}$ does not converge uniformly to g^2. [15]

(4) For $n \geq 1$ define $h_n : [-1,1] \to \mathbb{R}$ by
$$h_n(x) = \begin{cases} -1 & \text{if } \ x < \frac{-1}{n}; \\ nx & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n}; \\ 1 & \text{if } \frac{1}{n} < x. \end{cases}$$
Determine as to whether the family $\{h_n : n \geq 1\}$ is equi-continuous or not. Prove your claim. [15]

(5) Show that every non-principal ultra-filter on $\mathbb{N}$ contains the co-finite filter. Show that an ultra-filter on $\mathbb{N}$ is convergent iff it is a principal filter. [15]

(6) Let $\mathcal{F}$ be an ultra-filter on a set X and let $f : X \to Y$ be a function. Let $\mathcal{G}$ be the push-forward filter $f \star F$. If $V \in \mathcal{F}$, show that $W = \{f(v) : v \in V\}$ is in $\mathcal{G}$. [15]